Introduction to Geometric Algebra and its Applications to Physics

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HAPPJ 05.02.2026

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Space-Time Algebra

Second Edition

Birkhäuser

Outline

1. Geometric algebra
    a. Primitive objects
    b. Multiplying vectors
    c. Multivectors and Clifford algebra
    d. Multivectors as operations
2. Physical applications
    a. Algebra of space
    b. Algebra of spacetime
    c. Maxwell’s equations
    d. Dirac’s equation

Primitive geometric objects

1. scalar = “just magnitude”
2. vector = “magnitude + 1-direction”
3. bivector = “magnitude + 2-direction”

   …

4. k-vector = “magnitude + k-direction”

For vectors, position is irrelevant,

for k-vectors shape is irrelevant as well

Primitive geometric objects

- multiplication by scalar – just multiply magnitude!
- vector addition – parallelogram rule
- bivector addition:
  - if coplanar – just add magnitudes (noting orientation!)
  - if not – morph into parallelograms with shared side & direction cancelling out
  - then use 3D analog of parallelogram rule (“prism rule”?)
- in higher dimensions – hard to visualize and reason about
- can we get some algebraic way to manipulate k-vectors?

Multiplying vectors

$\vec{a}\vec{b}$ - what does it mean?

Let's discover together by requiring some basic decency from this product

$$(\alpha\vec{a} + \beta\vec{b})(\gamma\vec{c} + \delta\vec{d}) = \alpha\gamma\vec{a}\vec{c} + \beta\gamma\vec{b}\vec{c} + \alpha\delta\vec{a}\vec{d} + \beta\delta\vec{b}\vec{d}$$

$$\vec{a}\vec{a} = |\vec{a}|^2$$

Geometric product

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}$$

Symmetric part
(dot / inner product)

Antisymmetric part
(wedge / outer product)

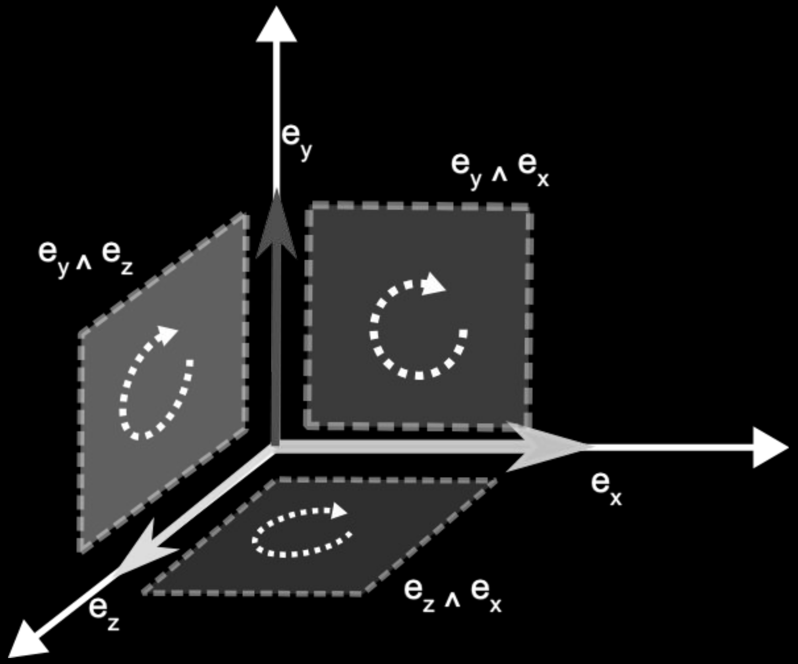
Product of vectors = scalar + bivector

Clifford algebra and multivectors

In N-dimensional physical space we define N orthonormal basis vectors:

$$e_i e_i = 1$$
$$e_i e_j = -e_i e_j \quad i \neq j$$

- All **k-**vectors can be expressed as products of basic vectors
- **Multi**vectors are linear combinations of k-vectors
- Formally, algebra of multivectors is known as **Clifford algebra**



2D application: rotations

- 2D bivector = imaginary unit
- scalar & bivector multivector = complex number
- product of two unit vectors = 2D rotation for angle between them
- inverse product = complex conjugate = inverted rotation

3D application: pseudo-things

- 3D trivector = pseudoscalar $\stackrel{?}{=}$ imaginary unit
- multiplication by i = correspondence between vectors and bivectors
- wedge product * i = cross product
- physics' pseudovectors (e.g. angular quantities & magnetic field) are actually bivectors!
- physics' pseudoscalars (magnetic flux & charge) are actually trivectors!

3D application: rotations

- rotation = automorphism that preserves vector norms
- rotation = two reflections
- rotor = product of even number of vectors
- rotation is done by sandwich operation
- angle doubling = isomorphism with spin group, which is a double cover of the underlying space

Algebra of space

- orthonormal basis in 3D
- trivector = pseudoscalar i (squares to -1 and commutes with everything)

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

×	σ_x	σ_y	σ_z
σ_x	I	$i\,\sigma_z$	$-i\,\sigma_y$
σ_y	$-i\,\sigma_z$	I	$i\,\sigma_x$
σ_z	$i\sigma_y$	$-i\,\sigma_x$	I

Spacetime algebra (STA)

orthonormal basis in spacetime consistent with Minkowski metric (+---):

$$\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \quad \mu \neq \nu$$

$$\gamma_0^2 = 1$$

$$\gamma_i^2 = -1 \quad i = 1, 2, 3$$

$$i = \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Spacetime -> space/time

- timelike = squares to 1
- spacelike = squares to -1
- … applies to all k-vectors, not just vectors

- conceptually, space should be “embedded” in spacetime
- is there such an algebraic mapping?

yes: **space-time split**

$$p\gamma_0 = p_0 + \mathbf{p}$$

Matryoshka of even subalgebras

```
Dirac algebra (16-dim = 1 s. + 4 v. + 6 bv. + 4 tv. + 1 ps.)
    -> Pauli algebra (8-dim = 1 s. + 3 v. + 3 bv. + 1 ps.)
        -> quaternions (4-dim)
            -> complex numbers (2-dim)
                -> real numbers (1-dim)
```

Space-time split and reference frames

- spacetime is done w.r.t. a particular time-like vector
- every time-like vector defines a reference frame!
- space-time split is a way to implicitly do Lorentz Transformation!

Lorentz Transformations in STA

- as is well-known, Lorentz boosts = hyperbolic rotation
- rotors in spacetime are defined similarly to 3D
- bivector defining plane of rotation can be split into space-like and time-like bivectors
- = every Lorentz rotation can be expressed as a special timelike rotation followed by a spatial rotation

$$p \to p' = e^{-\frac{1}{2} i \mathbf{b}} e^{-\frac{1}{2} \mathbf{a}} p \; e^{\frac{1}{2} \mathbf{a}} e^{\frac{1}{2} i \mathbf{b}}$$

a and **b** are **time-like bivectors** proportional to a selected γ_0

Multivector calculus in space & spacetime algebra

Pauli

$$\nabla = \sigma^k \partial_k$$

$$\nabla a = \nabla \cdot a + i\nabla \times a$$

gradient　　curl

Dirac

$$\Box \equiv \gamma^\mu \partial_\mu$$

Spacetime split

$$\gamma_0 \Box = \partial_0 + \boldsymbol{\nabla}$$

Physics example: Maxwell's equation(s)

$$\Box F = J.$$

$$F = \mathbf{E} + i\mathbf{B}$$

$$J = (J\gamma_0)\gamma_0 = (J \cdot \gamma_0 + J \wedge \gamma_0)\gamma_0 \equiv (\rho + \mathbf{J})\gamma_0.$$

$$\Box^2 F = \Box J = \Box \cdot J + \Box \wedge J.$$

$$\Box^2 F = \Box \wedge J$$

$$\Box \cdot J = 0 = \partial_0 \rho + \boldsymbol{\nabla} \cdot \mathbf{J}.$$

Physics example: vector potential

$$F = \Box A = \Box \cdot A + \Box \wedge A. \longrightarrow F = \Box \wedge A$$

$$\downarrow$$

$$\Box \cdot A = 0.$$

$$\Box^2 A = J.$$

Physics example: Dirac's equation (overview)

- ideal is a subalgebra invariant under multiplication by any element of the parent algebra
- Dirac bispinors are identified with elements of a minimal ideal in the Dirac algebra
- Dirac's equation can then be written

$$\Box\psi = (m + eA)\psi i.$$

Why should we care?

- coordinate-free formulation unifying simple geometry, relativistic physics, and even General Relativity
- Pauli and Dirac matrices reinterpreted as simply basis vectors in space and spacetime, with no connection to spin and particles
- geometric roots of imaginary unit in QM
- unified treatment of different transformations (classical rotations, relativistic boosts, gauge transformations) with rotors